KTH Computer Science
and Communication

2F1120 Spektrala transformer för Media
Solutions to Steiglitz, Chapter 1

Preface

This document contains solutions to selected problems from Ken Steiglitz's book: "A Digital Signal Processing Primer" published by Addison-Wesley. Refer to the book for the problem text.

1.5

The first reason is that our hearing is able to estimate the position of a sound source. If the two tuning forks are in different positions with respect of the listener, she might be able to discriminate between them.

Second, even if the forks are marked as being tuned to the same frequency, there might be small differences that can generate beats (see section 9 in the book).

Third: even if two objects are tuned to produce the same fundamental frequency, they can have different timbers depending on their shape/dimensions/material.

1.10

We look for the envelope of the expression

$$y(t) = a_1 \cos(\omega t) + a_2 \cos\big((\omega + \delta)t\big)$$

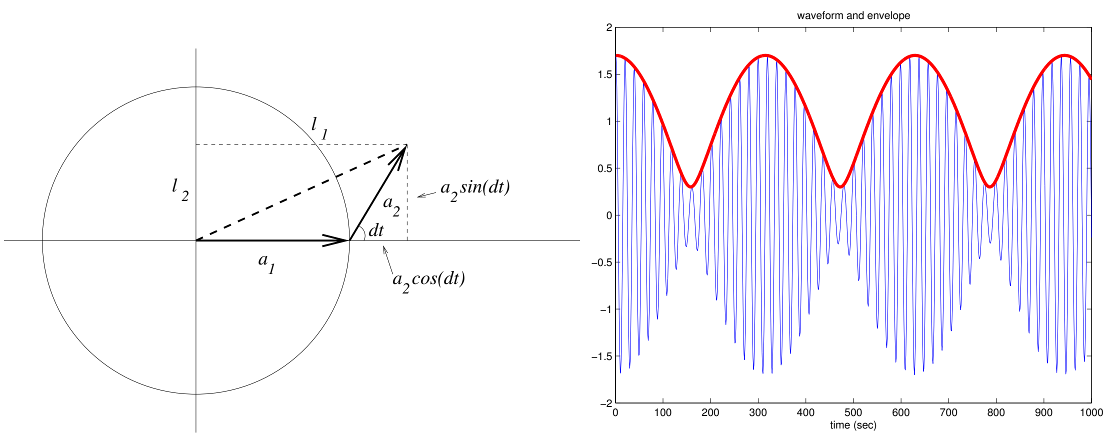

Figure 1. Schematic phasor representation (left) and waveform with envelope (right)

If we consider the phasor form of the expression 9.2 in the book:

$$a_1 e^{j\omega t} + a_2 e^{j(\omega+\delta)t}$$

and we plot it in Figure 1, we see that the envelope is the length of the vector obtained as the sum of the two terms. Using Pythagoras' theorem:

$$\begin{aligned} E(t) &= \sqrt{{l_1}^2 + {l_2}^2} = \sqrt{\big(a_1 + a_2\cos(\delta t)\big)^2 + {a_2}^2 \sin^2(\delta t)} \\ &= \sqrt{{a_1}^2 + \underbrace{{a_2}^2\cos^2(\delta t) + {a_2}^2\sin^2(\delta t)}_{{a_2}^2} + 2a_1 a_2 \cos(\delta t)} = \sqrt{{a_1}^2 + {a_2}^2 + 2a_1 a_2\cos(\delta t)} \end{aligned}$$

1.11

As can be seen in Figure 1 (right), for $t = 0$, for example, $\cos(\omega t) = \cos(\delta t) = \cos\big((\omega+\delta)t\big) = 1$, and

$$E(0) = \sqrt{{a_1}^2 + {a_2}^2 + 2a_1 a_2} = a_1 + a_2 = y(0)$$

proving that, at least in one point the envelope touches the curve.

1.12

The equation we search is obtained by imposing that $y(t) - E(t) = 0$, or, equivalently, $y(t) = E(t)$:

$$a_1\cos(\omega t) + a_2\cos\big((\omega+\delta)t\big) = \sqrt{{a_1}^2 + {a_2}^2 + 2a_1 a_2\cos(\delta t)}$$

Taking the square:

$$\begin{aligned} &{a_1}^2\cos^2(\omega t) + {a_2}^2\cos^2\big((\omega+\delta)t\big) + 2a_1 a_2\cos(\omega t)\cos\big((\omega+\delta)t\big) = \\ &\qquad = {a_1}^2 + {a_2}^2 + 2a_1 a_2\cos(\delta t) \end{aligned} \tag{1}$$

Grouping with respect to ${a_1}^2, {a_2}^2$ and $2a_1 a_2$,

$$\begin{aligned} &{a_1}^2\left[1 - \cos^2(\omega t)\right] + {a_2}^2\left[1 - \cos^2\big((\omega+\delta)t\big)\right] + \\ &\qquad + 2a_1 a_2\left[\cos(\delta t) - \cos(\omega t)\cos\big((\omega+\delta)t\big)\right] = 0 \end{aligned} \tag{2}$$

Now we recall that:

$$\cos^2(\alpha) + \sin^2(\alpha) = 1 \tag{3}$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \tag{4}$$

thus:

$$\begin{aligned} 1 - \cos^2(\omega t) &= \sin^2(\omega t) \\ 1 - \cos^2\big((\omega+\delta)t\big) &= \sin^2\big((\omega+\delta)t\big) \\ \cos(\omega t)\cos\big((\omega+\delta)t\big) &= \cos(\delta t) - \sin(\omega t)\sin\big((\omega+\delta)t\big) \end{aligned}$$

The first two are obtained from Equation 3, the last from Equation 4, imposing $\alpha = \omega t$ and $\beta = (\omega+\delta)t$. Substituting into Equation 2 we obtain:

$${a_1}^2\left[\sin^2(\omega t)\right] + {a_2}^2\left[\sin^2\big((\omega+\delta)t\big)\right] + 2a_1 a_2\left[\sin(\omega t)\sin\big((\omega+\delta)t\big)\right] = 0$$

That is clearly a square:

$$\left[a_1 \sin(\omega t) + a_2 \sin\big((\omega + \delta)t\big)\right]^2 = 0$$

That is equal to zero if and only if its argument is zero:

$$a_1 \sin(\omega t) + a_2 \sin\big((\omega + \delta)t\big) = 0$$

Note that, this equation is similar but not the same we would obtain by imposing that the first derivative of $y(t)$ be zero, indicating that the solutions are close, but not coincident with the maxima of $y(t)$. This equation does not have an analytic solution.

1.13

We want to find the analytical expression for the phase of the function $y(t)$. We can write the expression using phasors as in equation 9.3 in the book:

$$e^{j\omega t}\left[a_1 + a_2 e^{j\delta t}\right]$$

Then we recall that the phase of the product of two complex numbers is the sum of the respective phases:

$$\phi(t) = \angle\{e^{j\omega t}\} + \angle\{a_1 + a_2 e^{j\delta t}\}$$

The first term is just wt the second can be computed as the arctan of the ratio between the real and imaginary part of the expression:

$$\phi(t) = \omega t + \arctan\left(\frac{a_2 \sin(\delta t)}{a_1 + a_2 \cos(\delta t)}\right)$$

The instantaneous frequency is the derivative of this expression with respect to time:

$$F(t) = \frac{d\phi}{dt} = \omega + \frac{d}{dt} \arctan f(t) \tag{5}$$

where we named $f(t)$ the expression in parenthesis. We recall that the derivative of a composite function $g\left(f(x)\right)$ can be obtained as:

$$\frac{d}{dt} g\left(f(x)\right) = g'\left(f(x)\right) f'(x)$$

and that the derivative of $\arctan(x)$ is $\frac{1}{1+x^2}$. Then,

$$\frac{d}{dt} \arctan f(t) = \frac{f'(t)}{1 + f^2(t)}$$

Now we have to compute $f'(t)$: we call:

$$\begin{aligned} g(t) &= a_2 \sin(\delta t) \\ h(t) &= a_1 + a_2 \cos(\delta t) \end{aligned}$$

such that $f(t) = \frac{g(t)}{h(t)}$. Using the rule of derivative of a multiplication of functions:

$$f'(t) = \frac{d(g(t)/h(t))}{dt} = \frac{g'(t)h(t) - h'(t)g(t)}{h^2(t)}$$

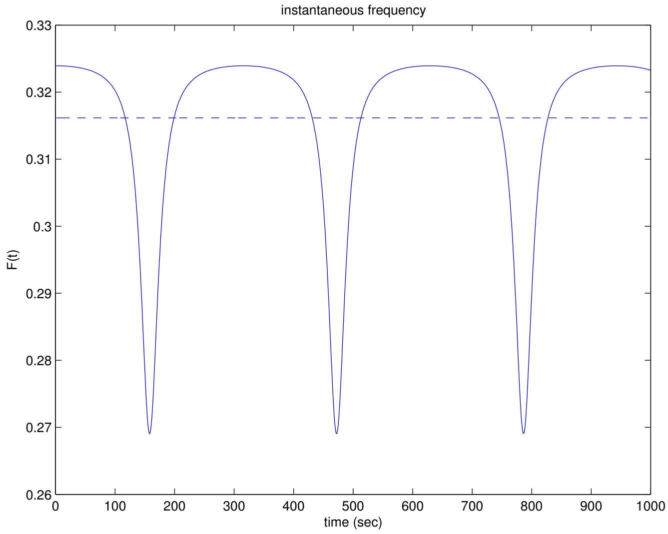

Figure 2. Plot of the solution for $\omega = 0.3157, a_1 = 1, a_2 = 0.7$ and $\delta = 0.02$. The dashed line is the average of the istantaneous frequency that is equal to ω

And finally:

$$\frac{f'(t)}{1+f^2(t)} = \frac{\frac{g'(t)h(t)-h'(t)g(t)}{h^2(t)}}{1+\frac{g^2(t)}{h^2(t)}} = \frac{\frac{g'(t)h(t)-h'(t)g(t)}{h^2(t)}}{\frac{h^2(t)+g^2(t)}{h^2(t)}} = \frac{g'(t)h(t)-h'(t)g(t)}{h^2(t)+g^2(t)}$$

with:

$$\begin{aligned} g'(t) &= a_2\delta\cos(\delta t) \\ h'(t) &= -a_2\delta\sin(\delta t) \end{aligned}$$

Substituting and simplifying (and adding the term ω from Equation 5):

$$F(t) = \omega + \frac{{a_2}^2\delta + a_1a_2\delta\cos(\delta t)}{{a_1}^2 + {a_2}^2 + 2a_1a_2\cos(\delta t)} \tag{6}$$

That is the solution to the problem. Figure 2 plots the function in Equation 6 for the values of the parameters specified in the text book at page 15.

1.17

The quantity δ in radiants per second (rad/s) corresponds to $2\pi f$ where f is the frequency in Hz, that is 1/s. The period of the oscillation is the inverse of the frequency:

$$T = \frac{1}{f} = \frac{2\pi}{\delta} \simeq \frac{6.28 \text{ rad}}{0.02 \text{ rad/s}} = 314 \text{ s}$$

1.18

We can solve the problem using phasors. Because we are considering a nonlinear operation, we have to be carefull. Infact if we have two sinusoidal signals $s_1(t) = a_1cos(\omega_1 t)$ and $s_2(t) =$

$a_2 cos(\omega_2 t)$, and we represent them with the corresponding phasors, we have:

$$\begin{aligned}
s_1(t) &= \Re\left\{a_1 e^{j\omega_1 t}\right\} \\
s_2(t) &= \Re\left\{a_2 e^{j\omega_2 t}\right\}
\end{aligned}$$

But, if $s(t)$ is the square of the sum of the two signals $s(t) = (s_1(t) + s_2(t))^2$, we cannot obtain this by first summing and squaring the phasors and then taking the real part. Infact:

$$\Re\left\{(x+y)^2\right\} \neq (\Re\{x\} + \Re\{y\})^2$$

One way to overcome this problem is to use Euler's formula:

$$\begin{aligned}
s_1(t) &= a_1 \left(\frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2}\right) \\
s_2(t) &= a_2 \left(\frac{e^{j\omega_2 t} + e^{-j\omega_2 t}}{2}\right)
\end{aligned}$$

Then the resulting singal is:

$$\begin{aligned}
s(t) &= \frac{{a_1}^2}{4}\left(e^{j\omega_1 t} + e^{-j\omega_1 t}\right)^2 + \frac{{a_2}^2}{4}\left(e^{j\omega_2 t} + e^{-j\omega_2 t}\right)^2 + \frac{2a_1 a_2}{4}\left(e^{j\omega_1 t} + e^{-j\omega_1 t}\right)\left(e^{j\omega_2 t} + e^{-j\omega_2 t}\right) \\
&= \frac{{a_1}^2}{4}\left(e^{j2\omega_1 t} + e^{-j2\omega_1 t} + 2\right) + \frac{{a_2}^2}{4}\left(e^{j2\omega_2 t} + e^{-j2\omega_2 t} + 2\right) + \\
&\quad \frac{a_1 a_2}{2}\left(e^{j(\omega_1+\omega_2)t} + e^{-j(\omega_1+\omega_2)t} + e^{j(\omega_1-\omega_2)t} + e^{-j(\omega_1-\omega_2)t}\right) \\
&= \frac{{a_1}^2}{2}\cos 2\omega_1 t + \frac{{a_2}^2}{2}\cos 2\omega_2 t + a_1 a_2 \cos(\omega_1+\omega_2)t + \\
&\quad a_1 a_2 \cos(\omega_1-\omega_2)t + \frac{{a_1}^2 + {a_2}^2}{2} \qquad (7)
\end{aligned}$$

Thus the frequencies are $2\omega_1$, $2\omega_2$, $\omega_1+\omega_2$, $\omega_1-\omega_2$, and the DC component with proportions indicated by the coefficients in Equation 7