KTH Computer Science
and Communication

2F1120 Spektrala transformer för Media
Solutions to Steiglitz Chapter 4

Preface

This document contains solutions to selected problems from Ken Steiglitz's book: "A Digital Signal Processing Primer" published by Addison-Wesley. Refer to the book for the problem text.

4.1

The three functions we want to sample are:

$$\begin{aligned} y_1(t) &= \sin(2\pi 0t + \phi) = \sin(\phi) \\ y_2(t) &= \sin(2\pi f_N t + \phi) \\ y_3(t) &= \sin(\pi f_N t + \phi) \end{aligned}$$

Note that $y_1(t)$ is a constant, in particular if the initial phase ϕ is zero, so is also $y_1(t)$. The samples are depicted in Figure 1. In the middle plot you can see that if the initial phase ϕ is zero, we sample the sinus in $y_2(t)$ always in the zero points. Finally $y_3(t)$ is depicted in the bottom plot.

Note that varying the initial phase ϕ, the results could be completely different: for example choosing $\phi = \pi/2$, the samples of the second function $y_2(t)$ are $+1$ and -1.

4.2

The filter

$$H(\omega) = 1 + a_1 e^{-j\omega\tau}$$

With $a_1 = 0.99$ and $\tau = 167\mu$sec reduces frequencies at odd multiples of 3kHz. This depends on the cosine term in equation 2.7 in the text book that we report here:

$$|H(\omega)| = \sqrt{1 + a_1^2 + 2a_1 \cos(\omega\tau)}$$

The cos term in the function varies between $+1$ and -1, assuming these values respectively for $\omega\tau = 2k\pi$ and $\omega\tau = (2k-1)\pi$. Consequently the function varies between $\sqrt{1 + a_1^2 + 2a_1} = \sqrt{(1 + a_1)^2} = |1 + a_1|$, and $\sqrt{1 + a_1^2 - 2a_1} = \sqrt{(1 - a_1)^2} = |1 - a_1|$. And the value $|1 + a_1|$ is assumed at $\omega\tau = 0, 2\pi$ while the value $|1 - a_1|$ is assumed at $\omega\tau = \pi$, in the range we are interested in. Now, if $a_1 > 0$, $|1 + a_1|$ is a maximum and $|1 - a_1|$ is a minimum, as for the continuous line in Figure 2. On the other hand, if $a_1 < 0$ maxima and minima are inverted, see dashed line in Figure 2. The solution to the problem is then obtained by choosing $a_1 = -0.99$.

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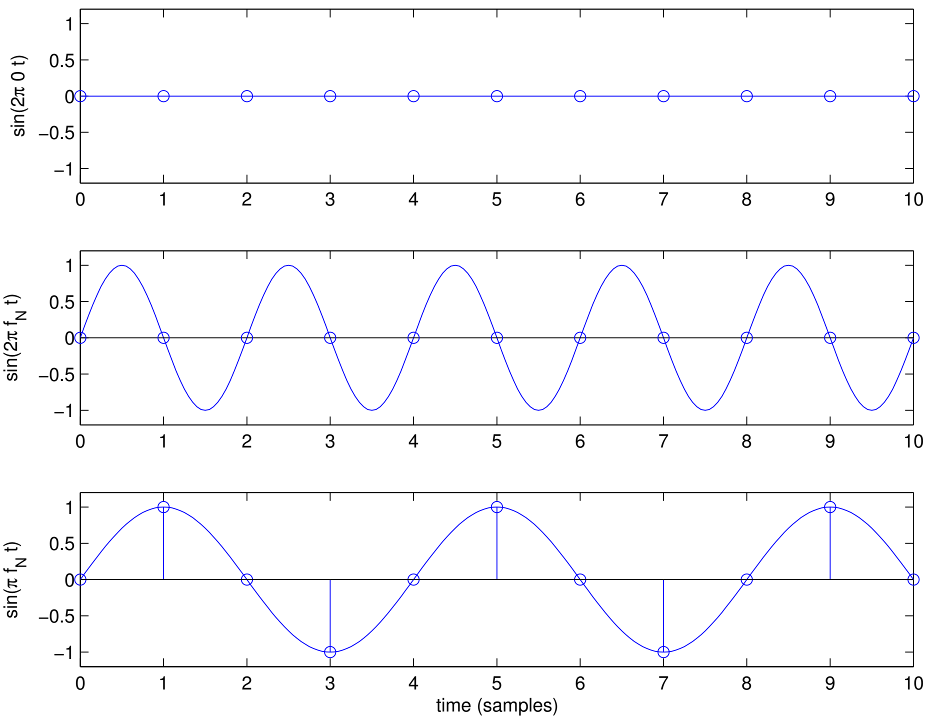


Figure 1.

4.3

The filter

$$y_t = x_t + x_{t-1} + x_{t-2} + \cdots + x_{t-99}$$

has transfer function

$$H(\omega) = \sum_{\tau=0}^{99} e^{-j\omega\tau}$$

Note that this is in the form

$$s_n = \sum_{\tau=0}^{n} a^\tau$$

We write down the first three terms of the partial sum s_n:

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 + a \\ s_2 &= 1 + a + a^2 \end{aligned}$$

Now we see that s_2 can be written in two different ways in terms of s_1:

$$\begin{aligned} s_2 &= s_1 + a^2 \\ s_2 &= 1 + a s_1 \end{aligned}$$

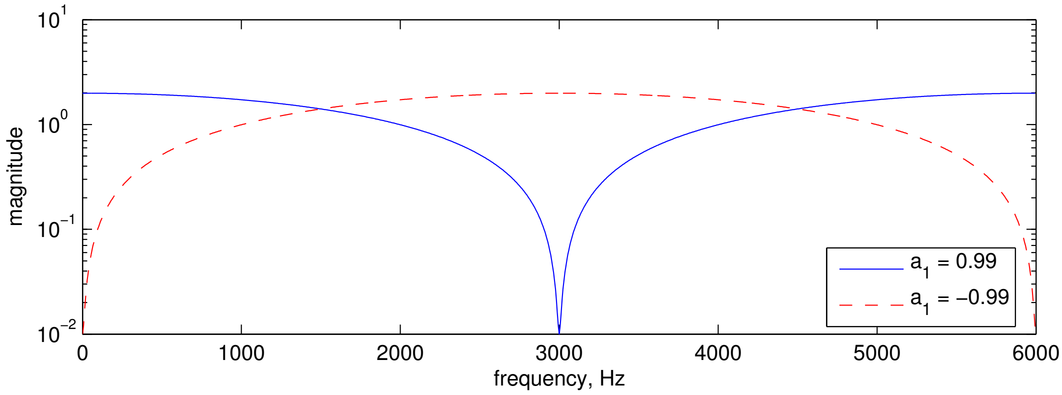


Figure 2.

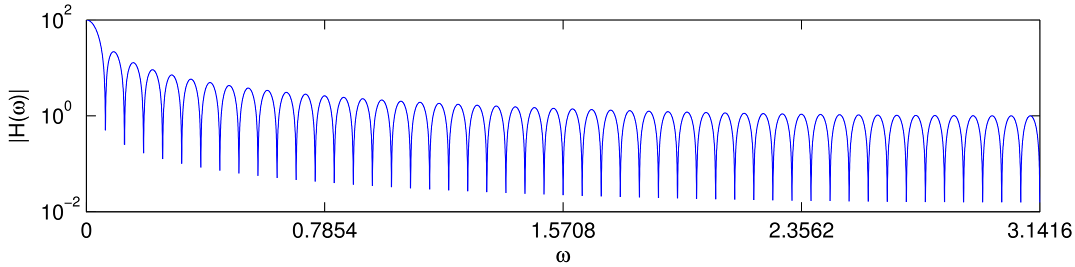


Figure 3.

And this is true more in general for any s_n in function of s_{n-1}:

$$\begin{aligned} s_n &= s_{n-1} + a^n \\ s_n &= 1 + a s_{n-1} \end{aligned}$$

Rearranging and substituting s_{n-1} from one equation to the other we obtain:

$$s_n = \frac{a^{n+1} - 1}{a - 1}$$

That in our case corresponds to:

$$H(\omega) = \frac{e^{-j100\omega} - 1}{e^{-j\omega} - 1}$$

Whose modulus is depicted in Figure 3. The zeros are for $e^{-j100\omega} = 1$, that is $100\omega = 2k\pi \iff \omega = k\pi/50$. The tops for $e^{-j100\omega} = -1$, that is $100\omega = (2k-1)\pi \iff \omega = (2k-1)\pi/50$

4.7

To suppress the DC component and $f_N/6$ we want a zero on $z = 1$ and two complex conjugate zeros in $z = e^{j\pi/6}$ and $z = e^{-j\pi/6}$. The transfer function is therefore of the kind:

$$\begin{aligned}
H(z) &= \frac{(1-z)}{z}\,\frac{(e^{j\pi/6}-z)}{z}\,\frac{(e^{-j\pi/6}-z)}{z} \\
&= \frac{(1-z)(1-z\overbrace{(e^{j\pi/6}+e^{-j\pi/6})}^{2\cos\pi/6=\sqrt{3}}+z^2)}{z^3} \\
&= \frac{1-\sqrt{3}z+z^2-z+\sqrt{3}z^2-z^3}{z^3} \\
&= -1+(1+\sqrt{3})z^{-1}-(1+\sqrt{3})z^{-2}+z^{-3}
\end{aligned}$$

The magnitude transfer function in ω can be written, starting from the first passage above as:

$$|H(\omega)| = \frac{|1-e^{j\omega}||e^{j\pi/6}-e^{j\omega}||e^{-j\pi/6}-e^{j\omega}|}{|e^{j3\omega}|} \tag{1}$$

The denominator $|e^{j3\omega}|$ is always equal to 1 and can be simplified. The square of the first term in the numerator in Equation 1 is:

$$\begin{aligned}
|1-e^{j\omega}|^2 &= \Re\left(1-e^{j\omega}\right)^2+\Im\left(1-e^{j\omega}\right)^2 \\
&= (1-\cos\omega)^2+\sin^2\omega \\
&= 1+\cos^2\omega-2\cos\omega+\sin^2\omega \\
&= 2(1-\cos\omega)
\end{aligned}$$

The square of the second term in Equation 1 is:

$$\begin{aligned}
|e^{j\pi/6}-e^{j\omega}|^2 &= (\underbrace{\cos\pi/6}_{\sqrt{3}/2}-\cos\omega)^2+(\underbrace{\sin\pi/6}_{1/2}-\sin\omega)^2 \\
&= 3/4+\cos^2\omega-\sqrt{3}\cos\omega+1/4+\sin^2\omega-\sin\omega \\
&= 2-\sqrt{3}\cos\omega-\sin\omega \qquad (2)
\end{aligned}$$

With the same procedure, but noticing that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ the square of the third term in Equation 1 is:

$$|e^{-j\pi/6}-e^{j\omega}|^2 = 2-\sqrt{3}\cos\omega+\sin\omega \tag{3}$$

Notice that Equations 2 and 3 are in the form $a+b$ and $a-b$ with $a = 2-\sqrt{3}\cos\omega$ and $b = \sin\omega$. Their product is thus a^2-b^2. Simplifying and inserting the three terms into Equation 1 we have:

$$\begin{aligned}
|H(\omega)|^2 &= 2(1-\cos\omega)(4\cos^2\omega-4\sqrt{3}\cos\omega+3) \\
&= 6-(8\sqrt{3}+6)\cos\omega+8(1+\sqrt{3})\cos^2\omega-8\cos^3\omega
\end{aligned}$$

Figure 4 shows the magnitude transfer function (top plot), the test signal (middle plot) and the output at regime (bottom plot). Note that the fluctuations in the output are due to numerical errors.

To find the maximum for $\omega \in (0, f_N/6)$ we note first that $|H(\omega)|$ is always positive. Since the square function is monotone for positive numbers, a maximum in $|H(\omega)|$ corresponds to a maximum in $|H(\omega)|^2$. Then it is sufficient to equal to zero the derivative of the function $|H(\omega)|^2$

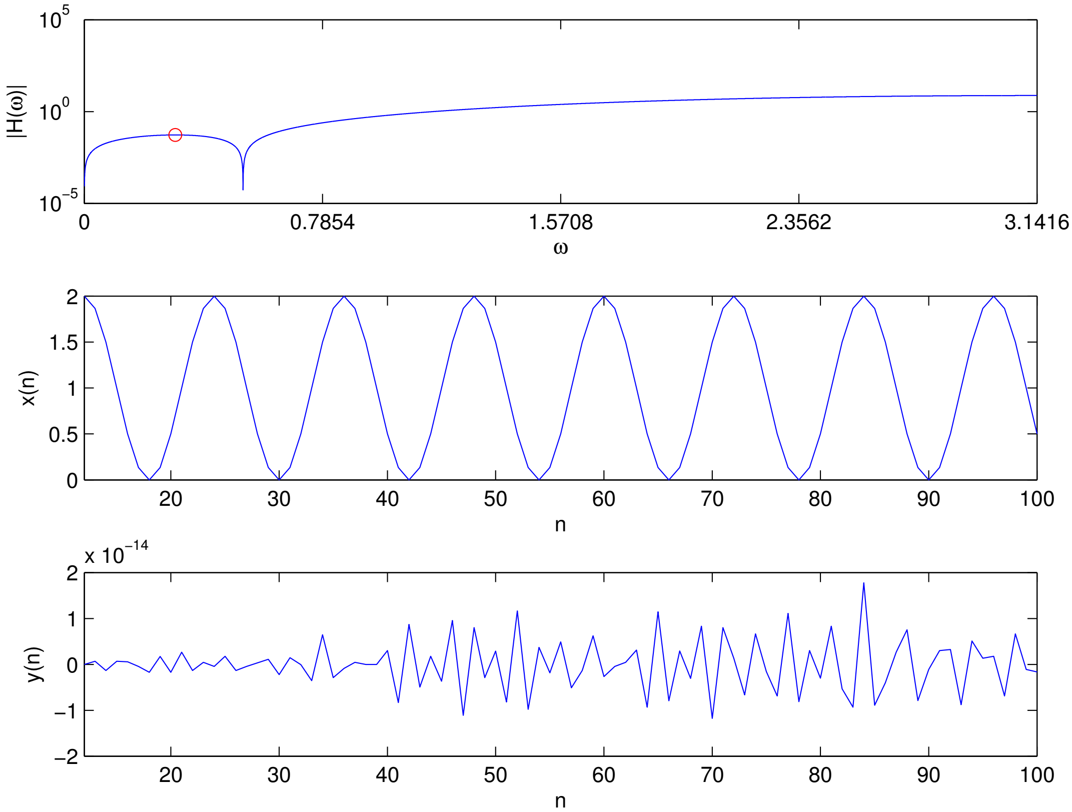
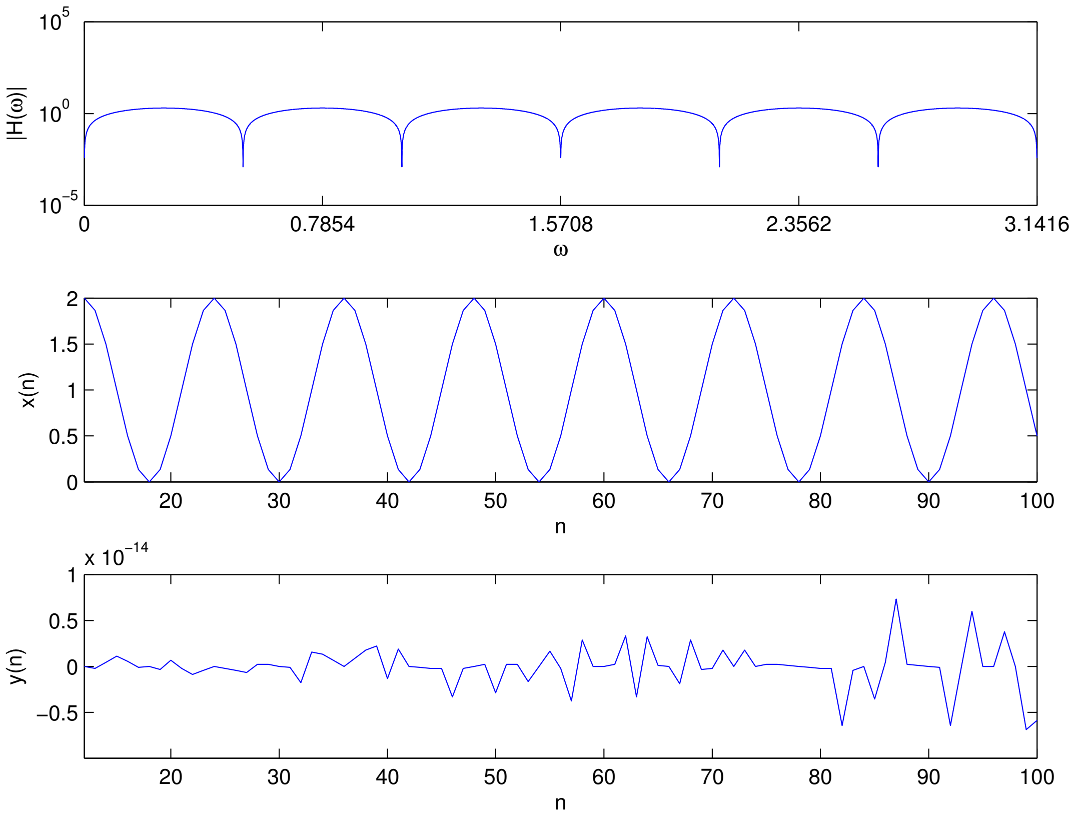

10^5
10^0
10^−5
|H(ω)|
0
0.7854
1.5708
2.3562
3.1416
ω
2
1.5
1
0.5
0
x(n)
20
30
40
50
60
70
80
90
100
n
x 10^−14
2
1
0
−1
−2
y(n)
20
30
40
50
60
70
80
90
100
n

Figure 4.



10^5
10^0
10^{-5}
$|H(\omega)|$
0
0.7854
1.5708
2.3562
3.1416
ω
x(n)
0
0.5
1
1.5
2
20
30
40
50
60
70
80
90
100
n
y(n)
x 10^{-14}
1
0.5
0
−0.5

Figure 5.

in function of the variable ω. To do this we remind that the derivatives of trigonometric functions are $\cos' x = -\sin x$ and $\sin' x = \cos x$, Then:

$$\begin{aligned}
A'(\omega) &= \frac{d|H(\omega)|^2}{d\omega} \\
&= (8\sqrt{3}+6)\sin\omega - 2\times 8(1+\sqrt{3})\cos\omega\sin\omega + 3\times 8\cos^2\omega\sin\omega \\
&= \sin\omega\left(24\cos^2\omega - 16(1+\sqrt{3})\cos\omega + 8\sqrt{3}+6\right)
\end{aligned}$$

When is this equal to zero? We notice that $\sin\omega$, $\omega \in [0, \pi/6]$ is zero only with $\omega = 0$ that is not an interesting solution. The second term is a quadratic form in $\cos\omega$. Solving with respect to $\cos\omega$ we obtain the two solutions:

$$\begin{aligned}
\cos\omega &= \frac{\sqrt{3}}{2} \\
\cos\omega &= \frac{2}{3} + \frac{\sqrt{3}}{6}
\end{aligned}$$

The first solution corresponds to the limit $\omega = \pi/6$. The second solution is the one we are interested in and corresponds to the peak between $[0, \pi/6]$, inverting we obtain:

$$\omega^* = \arccos\left(\frac{2}{3} + \frac{\sqrt{3}}{6}\right) \simeq 0.3 \text{ rad}$$

The corresponding value of the amplitude response is after simplification:

$$|H(\omega^*)| = \sqrt{\frac{104}{27} - \frac{20}{9}\sqrt{3}} \simeq 0.0534$$

Another way to solve the problem is to consider the inverse comb filter described in Section 8 in the text book. We see that to design a filter that stops the DC and an integer fraction of the Nyquist frequency it is sufficient to use the formula:

$$y_n = x_n + a x_{n-L}$$

To suppress the DC component we set a close to -1. To suppress $f_N/6$ we set $L = 12$. The resulting frequency response is depicted in Figure 5 (top plot) together with the example signal $x(n)$ (middle plot), and the output of the filter at regime (bottom plot). Note that the fluctuations are due to numerical errors

Which solutions is best depends on the desired behaviour of the filter in the rest of the base band.

4.9

The filter equation is

$$y_t = x_t + x_{t-1}$$

(a) If the signal is a ramp $x(t) = t$ the output will diverge at double the speed:

$$y_t = t + (t-1) = 2t - 1$$

See Figure 6 top left plot.

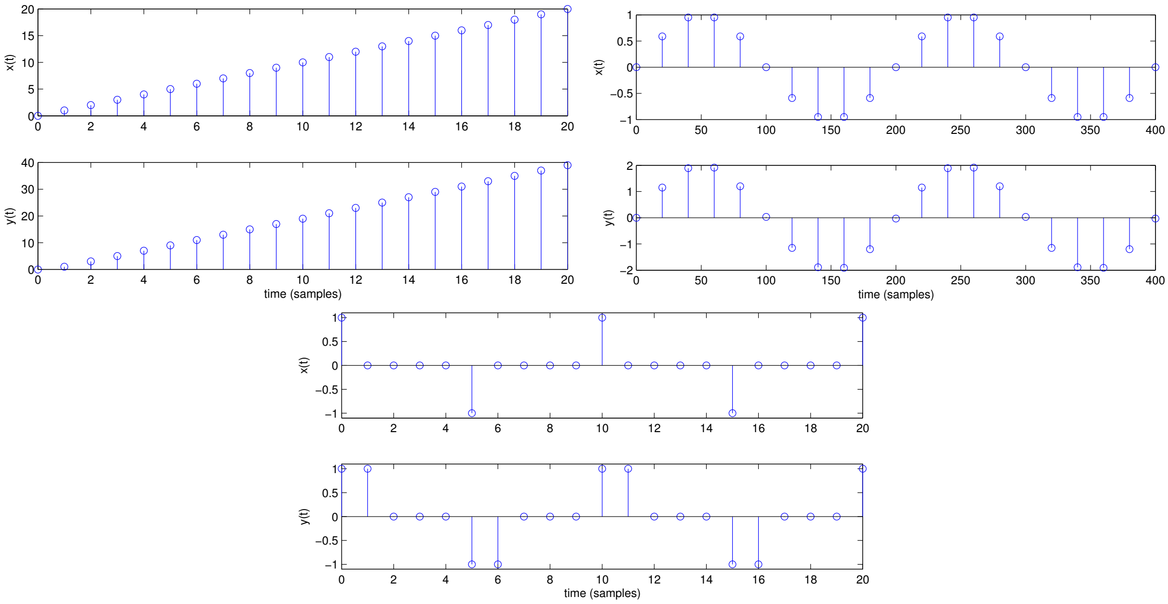


Figure 6.

(b) for the sinusoid $x(t) = \sin(\pi t/100)$:

$$y_t = \sin\left(\frac{\pi t}{100}\right) + \sin\left(\frac{\pi(t-1)}{100}\right) = 2\cos\left(\frac{\pi}{200}\right)\sin\left(\frac{\pi t}{100} - \frac{\pi}{200}\right) \tag{4}$$

The last equality is obtained by noting that

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\cos(\beta)\sin\alpha$$

and that the arguments to the two sines in Equation 4 can be written as

$$\begin{aligned} \frac{\pi t}{100} &= \frac{\pi t}{100} - \frac{\pi}{200} + \frac{\pi}{200} \\ \frac{\pi(t-1)}{100} &= \frac{\pi t}{100} - \frac{\pi}{200} - \frac{\pi}{200} \end{aligned}$$

and substituting

$$\begin{aligned} \alpha &= \frac{\pi t}{100} - \frac{\pi}{200} \\ \beta &= \frac{\pi}{200} \end{aligned}$$

See Figure 6 top right plot. Note that in the figure only every 20th sample is displayed for convenience.

(c) The definition given in the text book is probably wrong as t mod 5 is odd iff (if and only if) t is odd and even iff t even. There would not be any need to make use of the function mod if the author intended to define the function this way. Furthermore the function obtained by the definition is switching between -1 and $+1$ at every sample, which does not seem to be a very interesting example. We interpret the definition in the following way instead:

$$x(t) = \begin{cases} +1 & \text{if } t \text{ div } 5 \text{ is even and } t \text{ mod } 5 \text{ is zero} \\ -1 & \text{if } t \text{ div } 5 \text{ is odd and } t \text{ mod } 5 \text{ is zero} \\ 0 & \text{otherwise} \end{cases}$$

that is a signal with period ten samples, with a positive impulse in the beginning of each period and a negative impulse in the middle of each period (see Figure 6 bottom plot):

The output of the filter is

$$y(t) = \begin{cases} +1 & \text{if } t \text{ div } 5 \text{ is even and } t \bmod 5 \text{ is } 0 \text{ or } 1 \\ -1 & \text{if } t \text{ div } 5 \text{ is odd and } t \bmod 5 \text{ is } 0 \text{ or } 1 \\ 0 & \text{otherwise} \end{cases}$$

See Figure 6 bottom plot again.

If we compute the phase response $\phi(\omega) = \angle H(\omega)$ of the filter, we obtain, for each frequency, that is for each sinusoidal component, the *radian phase shift*. This is the rotation of each output phasor relative to the input phasor at frequency ω.

$$y(t) = A(\omega)\cos\big(\omega t + \phi(\omega)\big)$$

A more intuitive measure of the time shift is obtained by writhing the same equation as

$$y(t) = A(\omega)\cos\Big(\omega\big(t - \tau(\omega)\big)\Big)$$

$\tau(\omega)$ is called *phase delay* and its relation to the phase $\phi(\omega)$ is clearly:

$$\tau(\omega) = -\frac{\phi(\omega)}{\omega}$$

This gives the time delay in seconds of each sinusoidal component. For the filter in this example,

$$H(\omega) = 1 + e^{-j\omega T}$$

In the following, to make the result more general, we keep the T even if its value is equal to 1 sec in the example. The phase response is:

$$\begin{aligned}
\angle H(\omega) &= \arctan\left(\frac{\sin(-\omega T)}{1+\cos(-\omega T)}\right) \\
&= \arctan\left(-\frac{\sin(\omega T)}{1+\cos(\omega T)}\right) \\
&= -\arctan\left(\frac{2\sin(\omega T/2)\cos(\omega T/2)}{2\cos^2(\omega T/2)}\right) \\
&= -\arctan\left(\frac{\sin(\omega T/2)}{\cos(\omega T/2)}\right) == -\arctan\left(\tan\left(\frac{\omega T}{2}\right)\right) = -\frac{\omega T}{2}
\end{aligned}$$

(See Appendix A for an alternative way of solving the previous equation). And the corresponding phase delay is

$$\tau(\omega) = \frac{T}{2}$$

In this sense the filter gives a delay of half a sample. The same can be verified, for example in case (b), where the phase delay is explicit in the sinus term. In this case $T = 1$, $\omega = \pi/100$ and $\tau(\omega) = \frac{\pi}{200}/\frac{\pi}{100} = 1/2$

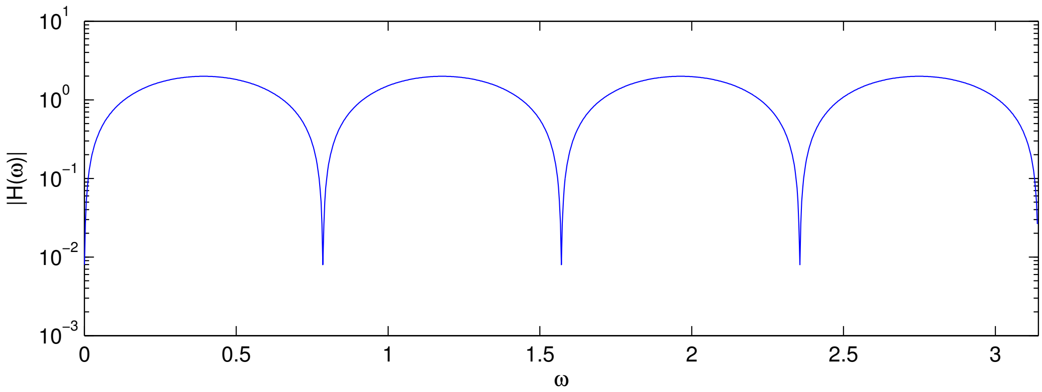


Figure 7.

4.11

The transfer function of the filter is:

$$H(\omega) = 1 - R^L e^{-j\omega L}$$

and its square modulus can be written as

$$\begin{aligned} |H(\omega)|^2 &= |1 - R^L e^{-j\omega L}|^2 &(5)\\ &= (1 - R^L \cos\omega L)^2 + R^{2L}\sin^2\omega L &(6)\\ &= 1 + R^{2L}\cos^2\omega L - 2R^L\cos\omega L + R^{2L}\sin^2\omega L &(7)\\ &= 1 + R^{2L} - 2R^L\cos\omega L &(8)\end{aligned}$$

When ω varies, the cosine assumes values between -1 and 1 and the amplitude response assumes respectively the maximum and minimum values:

$$\begin{aligned} \max|H(\omega)| &= \sqrt{1 + R^{2L} + 2R^L} = \sqrt{(1 + R^L)^2} = |1 + R^L| &(9)\\ \min|H(\omega)| &= \sqrt{1 + R^{2L} - 2R^L} = \sqrt{(1 - R^L)^2} = |1 - R^L| &(10)\end{aligned}$$

The whole function is plotted in Figure 7

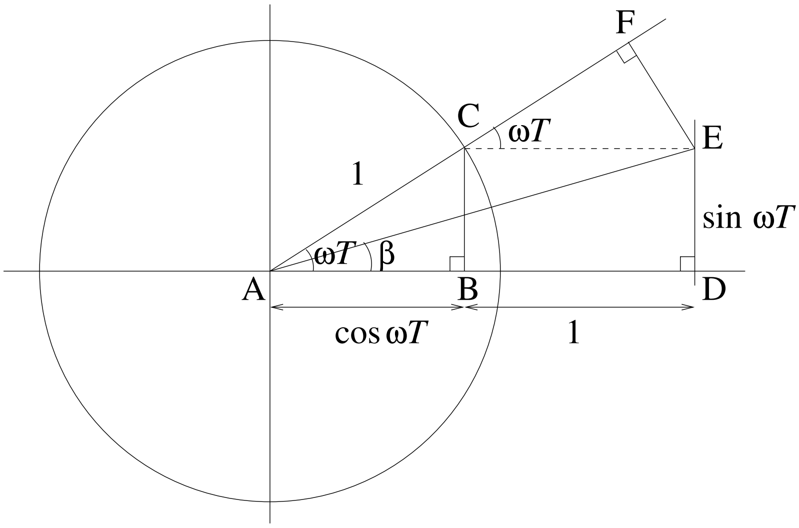


Figure 8.

A Computing the phase in ex. 4.9

We want to solve the equation

$$\angle H(\omega) = -\arctan \frac{\sin(\omega T)}{1 + \cos(\omega T)}$$

We would like the argument to the arctan function to be the tangent of some angle, in order to eliminate the functions. We refer to Figure 8. We construct the triangle ADE with side AD $= 1 + \cos \omega T$ and side DE $= \sin \omega T$. We call the angle EAD β. We note that the ratio between DE and AD is equal to the ratio between $\sin \beta$ and $\cos \beta$ and thus equal to $\tan \beta$. If we can find the angle β in function of ωT the problem is solved.

Now we look at triangle CEF. The angle ECF is equal to BAC (ωT) since the segment AB and CE are parallel to one another. The angle CFE is right ($\pi/2$) by construction, which means that angles CEF and ACB are also equal to guarantee that the sum of the interior angles of a triangle equal to π. Furthermore segment CE is long 1 because of how we built the figure. Triangles ABC and CEF are thus identical because they have the same interior angles and equal corresponding sides. We can deduct that segment EF is long $\sin \omega T$.

Now we move to the comparison between triangles ADE and AEF. These are right triangles with common hypotenuse, and sides DE and EF of equal length. This is enough to prove that ADE and AEF are also identical. The result is that angles DAE and EAF, that is β, are equal to $\omega T/2$.

Substituting we get

$$\angle H(\omega) = -\arctan\left(\tan\left(\frac{\omega T}{2}\right)\right) = -\frac{\omega T}{2}$$

that is the same solution we obtained previously applying trigonometric equations.