KTH Computer Science
and Communication

2F1120 Spektrala transformer för Media
Solutions to Steiglitz Chapter 5

Preface

This document contains solutions to selected problems from Ken Steiglitz's book: "A Digital Signal Processing Primer" published by Addison-Wesley. Refer to the book for the problem text.

5.3

(a) From the text book

$$\cos\theta_0 = \frac{1+R^2}{2R}\cos\theta$$

Figure 1 (left) plots θ_0 as a function of θ for different values of R. Note that the values have been normalised from 0 to 1. You can see that for $R = 1$, $\theta_0 = \theta$ in the whole range. When R approaches 0, the correspondence is less and less linear and in the limit θ_0 assumes only the two extreme values 0 and π. This corresponds to the frequency response having a maximum either at zero frequency or Nyquist frequency. On the right plot in Figure 1 the normalized amplitude

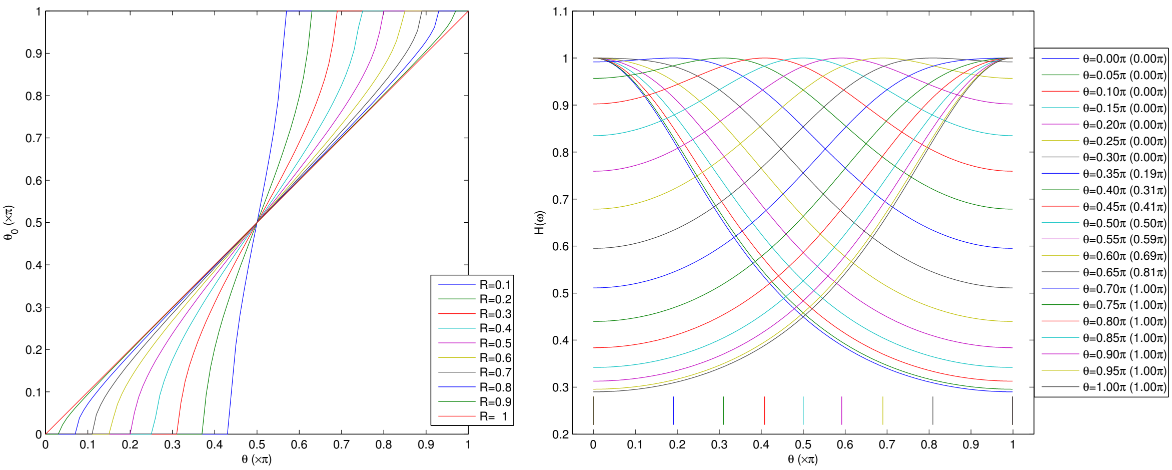

Figure 1.

response of the filter is plotted for $R = 0.3$ and for a number of values of θ (note that in order to simplify comparison, θ has been normalised and varies from 0 to 1 instead of 0 to π). The

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plot shows how for θ far from $\pi/2$ the peak of the spectrum is either at zero frequency or at the Nyquist frequency as noted before.
(b) As you can see in Figure 1 (left), when $\theta < \pi/2$, θ_0 is shifted to lower angles.

5.4

We need to compute equation 5.2 in the text book with $\phi = \theta$:

$$\begin{aligned}
\frac{1}{|H(\theta)|^2} &= (1-R^2)^2 + 4R^2\cos^2\theta - 4R(R^2+1)\cos^2\theta + 4R^2\cos^2\theta \\
&= (1-R^2)^2 - 4R(-2R+R^2+1)\cos^2\theta \\
&= (R-1)^2(R+1)^2 - 4R(R-1)^2\frac{1}{2}(\cos(2\theta)+1) \\
&= (R-1)^2[(R+1)^2 - 2R\cos(2\theta) - 2R] \\
&= (R-1)^2[R^2+1-2R\cos(2\theta)]
\end{aligned}$$

The gain is obtained taking the square root:

$$A = \frac{1}{|H(\theta)|} = (R-1)[R^2+1-2R\cos(2\theta)]^{\frac{1}{2}}$$

as we wanted to prove.

5.7

(a) The transfer function is obtained by multiplying at the denominator the terms in the form $1 - z_z z^{-1}$ for each zero z_z and at the denominator the terms in the form $1 - z_p z^{-1}$ for each pole z_p. In our case we have:

$$\begin{aligned}
H(z) &= \frac{(1-\sqrt{R}z^{-1})(1+\sqrt{R}z^{-1})}{(1-Re^{j\theta}z^{-1})(1-Re^{-j\theta}z^{-1})} \\
&= \frac{1-Rz^{-2}}{1-Rz^{-1}(e^{j\theta}+e^{-j\theta})+R^2z^{-2}} \\
&= \frac{1-Rz^{-2}}{1-2R\cos(\theta)z^{-1}+R^2z^{-2}} \qquad (1)
\end{aligned}$$

(b) If we write the transfer function as a fraction between the numerator $N(z)$ and the denominator $D(z)$ we obtain:

$$Y(z) = \frac{N(z)}{D(z)}X(z)$$

and rearranging the terms:

$$D(z)Y(z) = N(z)X(z)$$

that in our case is:

$$\left(1-2R\cos(\theta)z^{-1}+R^2z^{-2}\right)Y(z) = \left(1-Rz^{-2}\right)X(z)$$

In the time domain, each term z^{-1} corresponds to a delay, so we can write the same equation in time:

$$y_t - 2R\cos(\theta)y_{t-1} + R^2y_{t-2} = x_t - Rx_{t-1}$$

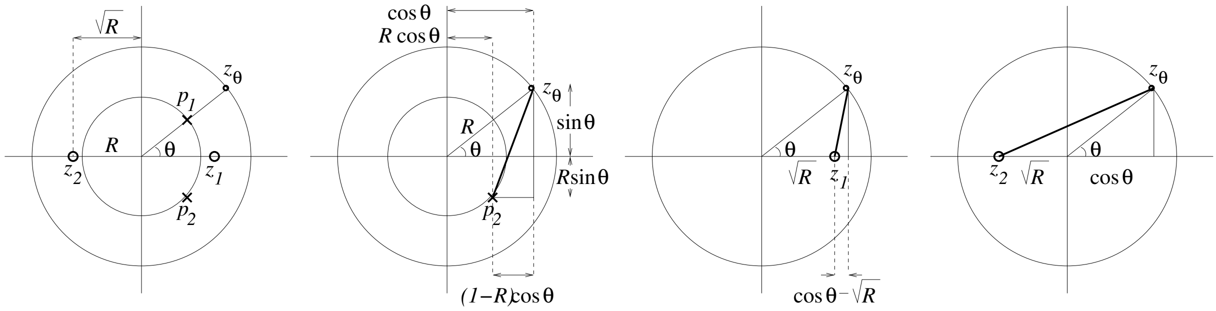

Figure 2.

moving all output terms beside the present to the right side:

$$y_t \quad = \quad x_t - Rx_{t-1} + 2R\cos(\theta)y_{t-1} - R^2 y_{t-2}$$

(c) A simple way to solve this task is to substitute the value $z = e^{j\theta}$ in Equation 1. Here we will show an alternative way using geometry. As depicted in Figure 2 the transfer function $H(z)$ has two poles in $p_{1,2} = Re^{\pm j\theta}$ and two zeros at $z_{1,2} = \pm\sqrt{R}$. Note that since $R < 1$, $\sqrt{R} > R$.

The amplitude response at angular frequency θ is given by the product of the distance between the point $z_\theta = e^{j\theta}$ and the zeros divided by the product of the distances from the poles[1], that is:

$$|H(\theta)| \quad = \quad \frac{|N(\theta)|}{|D(\theta)|} = \frac{|z_\theta - z_1||z_\theta - z_1|}{|z_\theta - p_1||z_\theta - p_2|}$$

Looking at the left plot in the figure, it is straightforward that:

$$|z_\theta - p_1|^2 = (1-R)^2$$

Looking at the second plot from the left we note that the square distance between p_2 and z_θ is the sum of the square lengths of the sides of the depicted triangle. The way to measure these lengths is indicated in the figure and leads to:

$$|z_\theta - p_2|^2 = (\cos\theta - R\cos\theta)^2 + (\sin\theta + R\sin\theta)^2 = (1-R)^2\cos^2\theta + (1+R)^2\sin^2\theta$$

From the third and fourth plots we see that:

$$\begin{aligned}
|z_\theta - z_1|^2 &= (\cos\theta - \sqrt{R})^2 + \sin^2\theta = 1 + R - 2\sqrt{R}\cos\theta \\
|z_\theta - z_1|^2 &= (\cos\theta + \sqrt{R})^2 + \sin^2\theta = 1 + R + 2\sqrt{R}\cos\theta
\end{aligned}$$

Noting that the last two terms are in the form $a - b$ and $a + b$, their product is $a^2 - b^2$, and the numerator is:

$$|N(\theta)|^2 \quad = \quad (1+R)^2 - 4R\cos^2\theta$$

For the denominator:

$$\begin{aligned}
|D(\theta)|^2 &= (1-R)^2\left[(1-R)^2\cos^2\theta + (1+R)^2\sin^2\theta\right] \\
&= (1-R)^2\left[(1+R^2-2R)\cos^2\theta + (1+R^2+2R)^2\sin^2\theta\right] \\
&= (1-R)^2[(1+R^2)\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + 2R\underbrace{(-\cos^2\theta + \sin^2\theta)}_{1-2\cos^2\theta}] \\
&= (1-R)^2\left[1 + R^2 + 2R - 4R\cos^2\theta\right] \\
&= (1-R)^2\left[(1+R)^2 - 4R\cos^2\theta\right]
\end{aligned}$$

[1] note that this is not inconsistent with what we wrote in exercise 5.7 as a term $1 - z_i z^{-1}$ is equivalent to $\frac{z - z_i}{z}$ and, the extra terms z do not affect the amplitude response: $|e^{j\omega}| = 1$

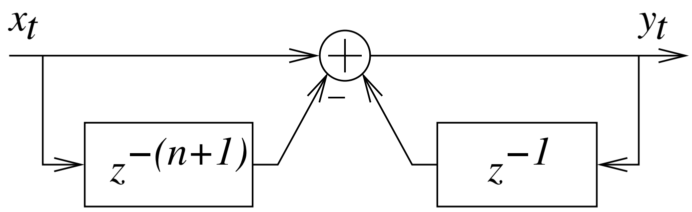

Figure 3.

And combining numerator and denominator:

$$
\begin{aligned}
|H(\theta)|^2 &= \frac{(1+R)^2 - 4R\cos^2\theta}{(1-R)^2\left[(1+R)^2 - 4R\cos^2\theta\right]} \\
&= \frac{1}{(1-R)^2}
\end{aligned}
$$

that is independent of θ as we wanted to prove.

5.9

We look for a feedback filter that has finite impulse response. We remind exercise 4.3 in Chapter 4 where the following filter was defined:

$$y_t = x_t + x_{t-1} + x_{t-2} + \cdots + x_{t-n}$$

where in that case $n = 99$. This filter is called *moving average*, because, besides a constant, it performs an average of the samples over a window that is shifted one sample at every time step. We know that this filter is FIR (Finite Impulse Response). In exercise 4.3 we obtained a simple analytic expression for the transfer function $H(\omega)$. Now we repeat the same derivation, but in the time domain. We write the output of the filter at time step $t-1$:

$$y_{t-1} = x_{t-1} + x_{t-2} + x_{t-3} + \cdots + x_{t-n-1}$$

We notice that the right term in the previous equation can be written in terms of y_t:

$$y_{t-1} = y_t - x_t + x_{t-(n+1)}$$

And rearranging:

$$y_t = x_t - x_{t-(n+1)} + y_{t-1}$$

Which can be implemented by the feedback filter in Figure 3.

An Infinite Impulse Response filter must be recursive (with feedback), if it was feed forward, in fact, each sample in the impulse response would be represented by a delay and a gain term. This would lead to an infinite number of elements in the filter, and to an infinite number of computations, which is not feasible.